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# On the Definition of Reducible Hypercomplex Number Systems.\*

BY SAUL EPSTEEN AND HEMAN BURR LEONARD.

§1.—Preliminary Remarks.

A hypercomplex number system is said to be reducible† when, by a proper choice of the units

$$E \equiv E_j E_k \equiv e_1 \cdot \dots \cdot e_m e_{m+1} \cdot \dots \cdot e_n$$
, with the relations  $e_{i_1} e_{i_2} = \sum_{i_1} \gamma_{i_1 i_2 i_3} e_{i_3}$ ,

the following conditions are fulfilled:

 $C_1$ ),  $E_i$  forms a system by itself,

$$e_{j_1}e_{j_2} = \sum_{j_2} \gamma_{j_1 j_2 j_3}e_{j_2}, \qquad (\gamma_{j_1 j_2 k} = 0);$$

 $C_2$ ),  $E_k$  forms a system by itself,

$$e_{k_1}e_{k_2} = \sum_{k_3} \gamma_{k_1 k_2 k_3} e_{k_3}, \qquad (\gamma_{k_1 k_2 j} = 0);$$
 $C_{jk}), \qquad e_j e_k = 0, \quad j = 1, \ldots, m, \qquad (\gamma_{jki} = 0);$ 
 $C_{kj}), \qquad e_k e_i = 0, \quad k = m + 1, \ldots, n, \quad (\gamma_{kji} = 0).$ 

The system E is supposed to be associative. We add moreover conditions concerning division.

- A), associativity;
- $C_r$ ), right-hand division possible and unique, that is, not every X is a right-hand divisor of zero;
- $C_l$ ), left-hand division possible and unique, that is, not every X is a left-hand divisor of zero.

<sup>\*</sup>Read before the Chicago Section of the American Mathematical Society, April 2, 1904.

<sup>†</sup> Benj. Peirce, American Journal of Mathematics, vol. 4 (1881), p. 100; Scheffers, Mathematische Annalen, vol. 39 (1891), p. 317.

In the previous paper of the same title\* it was shown that the conditions  $C_1$  and  $C_2$  are consequences of

(1) 
$$A$$
,  $C_{jk}$ ,  $C_{kj}$ ,  $C_r$ , or of

(2) A,  $C_{jk}$ ,  $C_{kj}$ ,  $C_l$ .

In (1) and (2) the assumptions A,  $C_r$ ,  $C_l$  are characterized by the fact that they refer to the system E as a whole.

Adopting suggestions of Professor E. H. Moore the associativity condition A is separated into eight parts and the conditions  $C_{ik}$  and  $C_{kj}$  are each separated into two parts.

The general number  $X_p = \sum_{i_1=1}^n x_{pi_1}e_{i_1}$  of the system E is the sum of two components

$$X_p = J_p + K_p = \sum_{j_1=1}^m x_{pj_1}e_{j_1} + \sum_{k_1=m+1}^n x_{pk_1}e_{k_1}.$$

The condition  $A: (X_1 X_2) X_3 = X_1 (X_2 X_3)$  is equivalent to the following eight:

$$\begin{array}{lll} A_1), & (J_1 J_2) J_3 &= J_1 (J_2 J_3); \\ A_2), & (K_1 K_2) K_3 = K_1 (K_2 K_3); \\ A_3), & (K_1 J_1) J_2 &= K_1 (J_1 J_2); \\ A_4), & (J_1 K_1) K_2 = J_1 (K_1 K_2); \\ A_5), & (J_1 K_1) J_2 = J_1 (K_1 J_2); \\ A_6), & (K_1 J_1) K_2 = K_1 (J_1 K_2); \\ A_7), & (J_1 J_2) K_1 = J_1 (J_2 K_1); \\ A_8), & (K_1 K_2) J_1 = K_1 (K_2 J_1). \end{array}$$

In general  $J_1 K_1 = J_2 + K_2$ . The condition  $C_{jk}$  says that  $\begin{cases} J_2 = 0 \\ K_2 = 0 \end{cases}$  simultaneously. Thus it is seen that  $C_{jk}$  is equivalent to the two conditions:

$$C_{jk}^{i}$$
),  $J_{1}K_{1} = 0 + K_{2} = K_{2}$ ;  $C_{jk}^{k}$ ),  $J_{1}K_{1} = J_{2} + 0 = J_{2}$ .

In general  $K_3J_3=J_4+K_4$ . Similarly then,  $C_{kj}$  says that  $\begin{cases} J_4=0 \\ K_4=0 \end{cases}$ 

<sup>#</sup>Epsteen, Transactions of the American Mathematical Society, vol. 5 (1904), p. 105.

simultaneously; and thus  $C_{kj}$  is equivalent to the two conditions:

$$C_{kj}^{j}$$
,  $K_{3}J_{3} = 0 + K_{4} = K_{4}$ ;  $C_{ki}^{k}$ ,  $K_{3}J_{3} = J_{4} + 0 = J_{4}$ .

$$C_{kj}^k$$
,  $K_3J_3 = J_4 + 0 = J_4$ .

The condition for right-hand division may be expressed thus:

 $C_r$ ), not every X is a right-hand divisor of zero; hence, an X exists such that

$$X_1 X = 0$$
, only if  $X_1 = 0$ .

The condition for left-hand division may be expressed thus:

 $C_i$ ), not every X is a left-hand divisor of zero; hence, an X exists such that

$$XX_1 = 0$$
, only if  $X_1 = 0$ .

When the condition  $C_1$  is not true, then there are at least two numbers of the set of units  $E_j$ , namely  $J_1$  and  $J_2$ , such that their product

$$J_1J_2 = J_3 + K_3 (K_3 \neq 0).$$

And likewise if  $C_2$  is not true, there are at least two numbers  $K_1$  and  $K_2$  of the set of units  $E_k$ , such that their product

$$K_1K_2 = J_3 + K_3$$
  $(J_3 \neq 0).$ 

The condition  $C_r^i$  which is employed in this paper means that

 $C_r^i$ ), not every J is a right-hand divisor of zero in the set  $E_i$ ; hence, a J exists such that

$$J_1 J = 0$$
, only if  $J_1 = 0$ .

Similarly  $C_i^j$  means that

 $C_i$ ), not every J is a left-hand divisor of zero in the set  $E_j$ ; hence, a J exists such that

$$JJ_1 = 0$$
, only if  $J_1 = 0$ .

From the preceding the meaning of  $C_r^k$  and  $C_l^k$  is evident.

We have, therefore, to consider in this paper the following twenty conditions:

- $A_1),$  $(J_1 J_2) J_3 = J_1 (J_2 J_3);$  $A_2$ ),  $(K_1 K_2) K_3 = K_1 (K_2 K_3);$  $(K_1 J_1) J_2 = K_1 (J_1 J_2);$  $A_3$ ),  $(J_1 K_1) K_2 = J_1 (K_1 K_2);$  $A_4$ ),  $A_5$ ),  $(J_1 K_1) J_2 = J_1 (K_1 J_2);$
- $(K_1 J_1) K_2 = K_1 (J_1 K_2);$  $A_6$ ),
- $(J_1 J_2) K_1 = J_1 (J_2 K_1);$  $A_7),$
- $(K_1 K_2) J_1 = K_1 (K_2 J_1);$  $A_8$ ),
- $C_1$ ),  $E_j$  is closed under multiplication, that is  $J_1J_2=J_3$ ;
- $(C_2)$ ,  $E_k$  is closed under multiplication, that is  $K_1K_2 = K_3$ ;
- $C_{ik}^{j}$ ),  $J_1K_1 = K_2 \qquad (J_2 = 0);$
- $J_1K_1=J_2$   $(K_2=0);$  $C_{ik}^k$ ),
- $K_1J_1 = K_2$   $(J_2 = 0);$   $K_1J_1 = J_2$   $(K_2 = 0);$  $C_{ki}^{i}$ ),
- $C_{ki}^k$ ),
- $C_r$ ), not every X is a right-hand divisor of zero;
- $C_l$ ), not every X is a left-hand divisor of zero;
- $C_r^i$ ), not every J is a right-hand divisor of zero in  $E_i$ ;
- $C_i^i$ ), not every J is a left-hand divisor of zero in  $E_i$ ;
- $C_r^k$ ), not every K is a right-hand divisor of zero in  $E_k$ ;
- $C_l^k$ ), not every K is a left-hand divisor of zero in  $E_k$ .

#### §2.—Dependencies.

We prove the following dependencies:

 $D_1$ ). The condition  $C_1$  says that  $J_1J_2=J_3$ . Suppose that

$$J_1J_2 = J_3 + K_3. (3)$$

Multiplying both sides on the right by K

$$(J_1J_2) K = J_3K + K_3K.$$

By  $A_7$ ,  $C_{jk}^j$ ,  $C_{jk}^k$  this becomes

$$0 = K_3 K.$$
By  $C_{kj}^i$ ,  $C_{kj}^k$  
$$0 = K_3 J.$$
Adding we obtain 
$$0 = K_3 (J + K) = K_3 X.$$

Hence by  $C_r$  $K_3=0$ .

The product  $J_1J_2$  being equal to  $J_3$  proves the condition  $C_1$ .

TABLE I.

Notation.		Assumptions.			C	onsequence	ce.	
$D_1$	$A_{7},\;C^{j}_{jk},$	$C^k_{jk},\;C^j_{kj},\;C^k_{kj},\;\;C^k_{kj}$	$\gamma_r$				$C_1$	
$D_2$	$A_3,\ C^j_{jk},$	$C^k_{jk},C^j_{kj},C^k_{kj},$	$C_{l}$				$C_1$	
$D_{3}$	$A_7,\ C^j_{jk},$	$C_{jk}^k$ ,	$C_{i}$	e			$C_1$	
$D_4$	$A_3$ ,	$C_{kj}^{i},\ C_{kj}^{k},$		$C_l^k$			$C_1$	
$D_{5}$	$A_8, C^j_{jk},$	$C^k_{jk},\;C^j_{kj},\;C^k_{kj}$ ,	$C_{r}$				$C_2$	
$D_{6}$	$A_4, C_{jk}^j,$	$C^k_{jk}$ , $C^i_{kj}$ , $C^k_{kj}$ ,	$C_l$				$C_{2}$	
$D_{7}$	$A_8$ ,	$C^j_{kj},\ C^k_{kj}$	$C_r^j$				$C_{2}$	
	$igg  A_4, \; C^j_{jk},$		$C_l^i$				$C_{2}$	
$D_{\mathfrak{g}}$	$A_5$ ,	$C^j_{kj},\ C^k_{kj},$	$C_r^{\jmath}$		$C^j_{jk}$			
$D_{10}$	$A_6$ ,	$C^j_{kj},\ C^k_{kj},$		$C_l^k$	$C^k_{jk}$			
$D_{11}$	$egin{aligned} A_6, \ C^{j}_{jk}, \ A_5, \ C^{j}_{jk}, \end{aligned}$	$O_{jk}^k$ ,	$C^{\kappa}_{r}$	;		$C_{kj}^k$		
$D_{12}$	$A_5, C^i_{ik},$	$C_{ik}^k$ ,	$C_{l}^{i}$			$C_{kj}^{j}$		

- $D_2$ ). Instead of multiplying (3) on the right by K and adding  $K_1J=0$ , we multiply it on the left by K and add  $JK_1=0$ . In this way it is easily seen that  $C_1$  is a consequence of  $A_3$ ,  $C_{jk}^i$ ,  $C_{jk}^k$ ,  $C_{kj}^i$ ,  $C_{kj}^k$ , and  $C_l$ .
- $D_3$ ). From equation (4) and  $C_r^k$  it follows that  $K_1 = 0$ . Hence  $C_1$  is a consequence of  $A_7$ ,  $C_{jk}^i$ ,  $C_{jk}^k$ ,  $C_r^k$ .
- $D_4$ ). From the equation  $KK_1 = 0$ , which arises in the proof of  $D_2$ , it follows from  $C_l^k$  that  $K_1 = 0$ . Hence  $C_1$  is a consequence of  $A_3$ ,  $C_{kj}^i$ ,  $C_{kj}^k$ ,  $C_l^k$ .
  - $D_{5}$ ). This follows by interchanging j and k in  $D_{1}$ .
  - $D_6$ ). This follows by interchanging j and k in  $D_2$ .
  - $D_7$ ). This follows by interchanging j and k in  $D_3$ .
  - $D_8$ ). This follows by interchanging j and k in  $D_4$ .

 $D_9$ ). The condition  $C^j_{jk}$  says that  $J_1K_1 = K_2$ . Suppose that  $J_1K_1 = J_2 + K_2$ . (5)

Multiplying both sides on the right by J

$$(J_1K_1)J = J_2J + K_2J.$$

By  $A_5$ ,  $C_{ki}^j$ ,  $C_{ki}^k$  this becomes

$$0 = J_2 J. (6)$$

By  $C_r^j$  we see that  $J_2 = 0$ . Therefore  $C_{jk}^j$  is a consequence of  $A_5$ ,  $C_{kj}^j$ ,  $C_{kj}^k$ , and  $C_r^j$ .

 $D_{10}$ ). Instead of multiplying (5) on the right by J, we multiply it on the left by K. In this way it is readily seen that  $C_{jk}^k$  is a consequence of  $A_6$ ,  $C_{kj}^i$ ,  $C_{kj}^k$ , and  $C_k^k$ .

 $D_{11}$ ). This follows by interchanging j and k in  $D_{9}$ .

 $D_{12}$ ). This follows by interchanging j and k in  $D_{10}$ .

The conditions  $C_1$ ,  $C_2$ ,  $C_{jk}^i$ ,  $C_{jk}^i$ ,  $C_{kj}^i$ ,  $C_{kj}^k$ , cannot be derived from any set of conditions other than the above.

#### §3.—Definitions of Reducibility by Independent Assumptions.

From Table I. it can be seen that there are seventy-eight different ways of defining the reducibility of a hypercomplex number system.

I.—The following eight definitions of reducibility are the only ones in which the division assumptions are on the set E as a whole.\*

<sup>\*</sup> All the definitions of this paper contain one or more division assumptions. There is only one definition, the classical one of Peirce-Scheffers, that does not contain a division assumption.

TABLE II.

(1)	(2)	(3)	Proved by (3)	(6)
Nota- tion.	From Table I.	Assumptions.	(4) (5)	Proved by (3) and (4).
$R_1$	1 1	$egin{aligned} A_4 , \ C^j_{jk} , \ C^k_{jk} , \ C^k_{kj} , \ C^k_{kj} , \ & C_l \ A_3 , \ C^j_{jk} , \ C^k_{jk} , \ C^k_{kj} , \ C^k_{kj} , \ & C_l \end{aligned}$	$egin{array}{c} C_{\mathbf{z}} \ C_{1} \end{array}$	
$R_2$	$D_6$	$egin{aligned} A_4,\; C^j_{jk},\; C^k_{jk},\; C^j_{kj},\; C^k_{kj},\; & C_l \ A_7,\; C^j_{jk},\; C^k_{jk},\; C^j_{kj},\; C^k_{kj},\; & C_r \end{aligned}$	$egin{array}{c} C_{2} \end{array}$	
$R_{ exttt{3}}$	$egin{bmatrix} D_6 \ \dots \end{bmatrix}$	$A_4,\ C^{i}_{jk},\ C^{k}_{jk},\ C^{i}_{jk},\ C^{i}_{kj},\ C^{k}_{kj},\ C_{1}$	$C_2$	
$R_4$	$egin{array}{c} D_5 \ D_2 \end{array}$	$egin{aligned} A_8, \ C^{j}_{jk}, \ C^{k}_{jk}, \ C^{j}_{jk}, \ C^{k}_{kj}, \ C^{k}_{kj}, \ A_3, \ C^{j}_{jk}, \ C^{k}_{jk}, \ C^{j}_{kj}, \ C^{k}_{kj}, \ C^{k}_{kj}, \ C^{k}_{lj}, \end{aligned}$	$egin{array}{c} C_{f 2} \\ C_{f 1} \end{array}$	
$R_{\scriptscriptstyle 5}$	$D_5$	$egin{aligned} A_8, & C^{i}_{jk}, & C^{k}_{jk}, & C^{i}_{kj}, & C^{k}_{kj}, & C_{r} \ A_7, & C^{i}_{jk}, & C^{k}_{jk}, & C^{i}_{kj}, & C^{k}_{kj}, & C_{r} \end{aligned}$	$egin{array}{c} C_{f 2} \\ C_{f 1} \end{array}$	
$R_{6}$	1 1	$A_{8},\;C_{jk}^{j},\;C_{jk}^{k},\;C_{kj}^{g},\;C_{kj}^{k},\;$	$C_2$	
$R_{7}$	$D_2$		$C_1$	
$R_{ m s}$	$D_1$	$egin{aligned} A_3, \ C^{i}_{jk}, \ C^{k}_{jk}, \ C^{k}_{jj}, \ C^{k}_{kj}, \ C^{k}_{kj}, \ C_2 \ & C_2 \ & C_4 \ & C_7, \ C^{i}_{jk}, \ C^{i}_{jk}, \ C^{i}_{kj}, \ C^{k}_{kj}, \ C^{k}_{kj}, \ C_2 \ & C_3 \ & C_4 \ & C_6 \ & C_7 \ & C_8 \ & C_8$	$C_1$	

II.—The following thirty-eight definitions of reducibility are definitions in which the division assumptions are solely on the subsets  $E_j$ ,  $E_k$ .

TABLE III.

(1)	(2)	(	(3)	Proved b	оу (3)	(6)
Nota- tion.	From Table I.	Assur	nptions.	(4)	(5)	Proved by (3) and (4).
	$D_{12}$	$A_{5},~C^{j}_{jk},~C^{k}_{jk},$	Cį	$C^j_{k\!j}$		
70	$D_{11}$	$A_6$ , $C^j_{jk}$ , $C^k_{jk}$ ,	$C_r^k$	$C^k_{k\!j}$		
$R_{9}$	$D_8$	$A_4$ , $C^j_{jk}$ , $C^k_{jk}$ ,	$C_l^g$		$C_{2}$	
	$D_4$	$A_3$ ,	$C_l^k$	" "		$C_1$
	$D_{12}$	$A_5,\ C^i_{jk},\ C^k_{jk},$	$C_{i}$	$C^{j}_{kj} \ C^{k}_{kj}$		
$R_{10}$	$D_{11}$	$A_6$ , $C^j_{jk}$ , $C^k_{jk}$ ,	$C_r^k$	$C_{\it kj}^{\it k}$		
10	$D_8$	$A_4$ , $C^j_{jk}$ , $C^k_{jk}$ ,	$C_l^i$	1	$\alpha$	
	$D_3$	$A_7, C^j_{jk}, C^k_{jk},$	$C_{r}^{\prime k}$	$C^{j}_{k j} \ C^{k}_{k j}$	$C_{1}$	
	$oxed{D_{12}}$	$A_5,\ C^j_{jk},\ C^k_{jk},$	$C_{l}^{g}$	$C^{j}_{k\!j}$		
$R_{11}$	$D_{11}$	$A_6, C^j_{jk}, C^k_{jk},$	$C^k_r$	$C_{k\!j}^k$		
	$D_8$	$A_4$ , $C^j_{jk}$ , $C^k_{jk}$ ,	$C_l^i$		$C_{2}$	
			$C_{\mathbf{i}}$			
	$D_{12}$	$A_5,\ C^j_{jk},\ C^k_{jk},$	$C_{l}^{g}$	$C^{j}_{k\! j}$		
$R_{12}$	$D_{11}$	$A_6$ , $C^j_{jk}$ , $C^k_{jk}$ ,	$C^{k}_{r}$	$C_{kj}^k$		
	$D_7$	$A_{8}$ ,	$C_{r}^{j}$	" "		$C_{2}$
	$D_4$	$A_3$ ,	$C_l^k$	" "		$C_1$
	$D_{12}$	$A_{6}, C^{j}_{jk}, C^{k}_{jk},$ $A_{4}, C^{j}_{jk}, C^{k}_{jk},$ $A_{3},$ $A_{5}, C^{j}_{jk}, C^{k}_{jk},$ $A_{6}, C^{j}_{jk}, C^{k}_{jk},$ $A_{7}, C^{j}_{jk}, C^{k}_{jk},$ $A_{8}, C^{j}_{jk}, C^{k}_{jk},$ $A_{8}, C^{j}_{jk}, C^{k}_{jk},$ $A_{8}, C^{j}_{jk}, C^{k}_{jk},$ $A_{5}, C^{j}_{jk}, C^{k}_{jk},$ $A_{6}, C^{j}_{jk}, C^{k}_{jk},$ $A_{6}, C^{j}_{jk}, C^{k}_{jk},$ $A_{6}, C^{j}_{jk}, C^{k}_{jk},$ $A_{6}, C^{j}_{jk}, C^{k}_{jk},$ $A_{8},$ $A_{7}, C^{j}_{jk}, C^{k}_{jk},$ $A_{8},$ $A_{7}, C^{j}_{jk}, C^{k}_{jk},$ $A_{8},$ $A_{7}, C^{j}_{jk}, C^{k}_{jk},$	$C_{l}^{j}$	$C_{kj}^{j}$		
$R_{12}$	$D_{11}$	$A_6$ , $C^j_{jk}$ , $C^k_{jk}$ ,	$C_r^k$	$C_{kj}^k$		
19	$D_7$	$A_8$ ,	$C^{j}_{r}$	" "		$C_{2}$
	$D_3$	$A_7, C^j_{jk}, C^k_{jk},$	$C^k_r$		$C_1$	

TABLE III.—Continued.

(1) Nota-	(2) From		(3)	again, an agus a sa ann ann ann an an an an an aige ann an gan an agus an an agus an an agus an agus an agus a	Proved b	y (3)	(6) Proved
tion.	Table I.		Assumptions.		(4)	(5)	by (3) and (4).
	1	$A_5$ , $C^j_{jk}$ , $C^k_{jk}$ ,		$C_l^j$	$C^{j}_{kj}$		
$R_{14}$	$D_{11}$	$A_6, C^j_{jk}, C^k_{jk},$		$C_r^k$	$C_{kj}^k$		
	$oxed{D_3}$	$A_{7},\ C^{j}_{jk},\ C^{k}_{jk},$ $A_{5},\ C^{j}_{jk},\ C^{k}_{jk},$	$C_{2}$	$C_r^k$		$C_1$	
	$D_{12}$	$A_{\mathfrak{z}},\; C^{j}_{jk},\; C^{k}_{jk},$		$C_l^i$	$C^{j}_{kj}$		
$R_{ m 15}$	$D_{11}$	$A_6,\ C^j_{jk},\ C^k_{jk},$	$C_1$	$C_r^k$	$C_{kj}^k$		
			$C_{2}$				
	$D_{12}$	$A_{\mathfrak{z}},\ C^{j}_{jk},\ C^{k}_{jk},$		$C_l^j$	$C^{j}_{kj}$	$C_{2}$	
$R_{16}$	$D_8$	$A_4, C^j_{jk}, C^k_{jk},$		$C_l^j$		$C_1$	
	$egin{array}{c} D_3 \ \dots \end{array}$	$egin{aligned} A_5, & C^j_{jk}, & C^k_{jk}, \ A_4, & C^j_{jk}, & C^k_{jk}, \ A_7, & C^j_{jk}, & C^k_{jk}, \ A_5, & C^j_{jk}, & C^k_{jk}, \ A_4, & C^j_{jk}, & C^k_{jk}, \ \end{aligned}$	$C^k_{kj}$	$C^k_{r}$		91	
	$D_{12}$	$A_5,\ C^j_{jk},\ C^k_{jk},$		$C_l^j$	$C^{j}_{kj}$		
$R_{17}$	$D_8$	$A_4$ , $C^j_{jk}$ , $C^k_{jk}$ ,	$C_{kj}^k$	$C_l^j$		$C_2$	
			$C_{\mathbf{i}}$				
	$D_{12}$	$A_5,\ C^j_{jk},\ C^k_{jk},$		$C_l^j$	$C^{j}_{kj}$		
$R_{ m 18}$	$D_3$	$A_7$ , $C^j_{jk}$ , $C^k_{jk}$ ,	$C^k_{kj}$	$C^k_{r}$		$C_1$	
			$C_{oldsymbol{2}}$				
	$D_{12}$	$A_{\mathfrak{z}},\ C^{j}_{jk},\ C^{k}_{jk},$	$C^k$	$C_l^i$	$C^{ij}_{kj}$		
$R_{ m 19}$			$egin{array}{ccc} C_{kj}^k & & & & & & & & & & & & & & & & & & &$				
			$C_{\mathbf{z}}$				
	$D_{11}$	$egin{aligned} A_6, \ C^j_{jk}, \ C^k_{jk}, \ A_4, \ C^j_{jk}, \ C^k_{jk}, \ A_7, \ C^j_{jk}, \ C^k_{jk}, \end{aligned}$		$C^k_r$	$C_{kj}^k$		
$R_{20}$	$D_8$	$egin{array}{cccccccccccccccccccccccccccccccccccc$		$egin{aligned} C_l^g \ C_r^k \end{aligned}$		$C_1$	
	$ig _{\cdots}^{D_3}$	$A_7,                   $	$C_{kj}^{\ j}$	····		<b>∪</b> 1	

TABLE III.—Continued.

(1)	(2)	(3)		Proved by	y (3)	(6)
Nota- tion.	From Table I.	Assumptions.		(4)	(5)	Proved by (3) and (4).
$R_{21}$	$D_8 \ \cdots \ \cdots$	$A_6,\ C^i_{jk},\ C^k_{jk},\ A_4,\ C^i_{jk},\ C^k_{jk},\ C^k_{jk},\ C^k_{kj}$	$C_r^k$	$C_{kj}^k$	$C_2$	
$R_{22}$	$egin{array}{c} D_{11} \ D_3 \ \dots \end{array}$	$egin{aligned} A_{6}, \ C^{i}_{jk}, \ C^{k}_{jk}, \ C^{i}_{jk}, \ C^{i}_{jk}, \ C^{i}_{kj} \ & C_{2} \ & & C^{i}_{kj} \ & C$	$rac{C_{r}^{k}}{C_{r}^{k}}$	$C_{kj}^k$	$C_1$	
		$C_{2}$	$C^k_r$	$C_{kj}^k$		
$R_{24}$	$egin{array}{c} D_{10} \ D_{9} \ D_{8} \ D_{4} \ \end{array}$	$egin{array}{lll} A_6, & C_{kj}^i, \ C_{kj}^k, \ C_{kj}^j, \ C_{kj}^k, \ C_{kj}^i, \ C_{kj}^k, \ \end{array} \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ $	$C^k$ $C^j_r$ $C^i_l$ $C^k_l$	$C^k_{jk}$ $C^c_{jk}$ $\cdots$ $\cdots$	$oxed{C_1}$	$C_2$
$R_{25}$	$egin{array}{c} D_{10} \\ D_{9} \\ D_{8} \\ D_{3} \end{array}$	$egin{array}{lll} A_6, & & C^{j}_{kj}, \ C^{k}_{kj}, \ & A_5, & & C^{j}_{kj}, \ C^{k}_{kj}, \ & A_4, \ & A_7, \end{array}$	$C_l^k \ C_r^j \ C_r^k$	$C^k_{jk}$ $C^j_{jk}$ $\cdots$ $\cdots$		$egin{array}{cccc} & & & & & & & & & & & & & & & & & $
$R_{26}$	$egin{array}{c c} D_{10} \\ D_{9} \\ D_{7} \\ D_{4} \\ \end{array}$	$egin{array}{lll} A_6, & C_{kj}^i, \ C_{kj}^k, \ C_{kj}^i, \ C_{kj}^k, \ C_{kj}^i, \ C_{kj}^k, \ C_{kj}^i, \ C_{kj}^k, \ C_{kj}^i, \ C_{$	$C_l^k$ $C_r^j$ $C_r^j$ $C_l^k$	$C^k_{jk} \ C^j_{jk}$	$oxed{C_2}$	

TABLE III.—Continued.

(1)	(2)		(3)			Proved	by (3)	(6)
Nota- tion.	From Table I.		Assumptions.			(4)	(5)	$\begin{array}{c} \textbf{Proved} \\ \textbf{by} \\ (3) \textbf{ and } (4). \end{array}$
	$D_{10}$	$A_6$ ,	$C^j_{kj},\ C^k_{kj},$		$C_l^k$	$C^k_{jk}$		
$R_{or}$	$egin{array}{c} D_9 \ D_7 \ D_3 \ \end{array}$	$A_5$ ,	$C_{kj}^j,\; C_{kj}^k,$	$C_r^j$		$C^{j}_{jk}$		
21	$D_7$	$A_8$ ,	$C^j_{kj},\;C^k_{kj},$	$C_r^j$			$C_2$	
	$D_3$	$A_7$ ,			$C^k_r$	"		$C_1$
	$D_{10}$	$A_6$ ,	$C^j_{kj},\ C^k_{kj},$		$m{C}_l^k$	$C^k_{jk}$		
R	$D_9$	$A_5$ ,	$C^j_{kj},\ C^k_{kj},$	$C_r^j$		$C^{j}_{jk}$		
1028	$egin{array}{c} D_{10} \ D_{9} \ D_{7} \ \dots \end{array}$	$A_8$ ,	$C_{kj}^{j},\;C_{kj}^{k},\ C_{1}$	$C_r^j$			$C_2$	
	$D_{10}$	$A_{\epsilon},$	$C_{kj}^j,\;C_{kj}^k,$		$C_l^k$	$C^k_{jk}$		
$\mathcal{D}$	$D_9$	$A_5$ ,	$C^j_{kj},\ C^k_{kj},$	$C_r^j$		$C^j_{jk}$		
1029	$D_4$	$egin{aligned} A_6,\ A_5,\ A_3, \end{aligned}$	$C^j_{kj},\ C^k_{kj},$ $C_2$		$C_l^k$		$C_1$	
	$D_{10}$ $D_{9}$ $\dots$	$A_{6}$ ,	$C^j$ . $C^k$ .		$C_l^k$	$C^k_{jk}$		
$R_{ m 30}$	$D_9$	$A_5$ ,	$C^j_{kj},\;C^k_{kj}, \ C.$	$C_{r}^{j}$		$igg  C^j_{jk}$		
			U <sub>0</sub>					
	$D_{10}$	$A_6$ ,	$C^j_{kj}$ , $C^k_{kj}$ ,		$C_l^k$	$C^k_{jk}$		
$R_{a_1}$	$D_7$	$A_8$ ,	$C^j_{kj},\ C^k_{kj},$	$C^{j}_{m{r}}$			$C_2$	
2031	$D_4$	$igg  A_3$ , $C^j_{ik}$	$egin{array}{cccccccccccccccccccccccccccccccccccc$		$C_l^k$		$C_1$	
	$D_{10}$	$A_6$ ,	$C^j_{kj},\ C^k_{kj},$		$C_l^k$	$C^k_{jk}$		
$R_{32}$	$D_7$	$A_8$ ,	$C_{kj}^j,\ C_{kj}^k,$	$C_r^{j}$			$C_2$	
	<u> </u>	$igcup_{jk}$	$C_1$					

TABLE III.—Continued.

(1)	(2)		(3)				Proved	by (3)	(6)
Nota- tion.	From Table I.		Assumpt	ions.			(4)	(5)	Proved by (3) and (4).
	$D_{10}$	$A_6$ ,	$C^j_{kj},\ C^k_{kj},$			$C_l^k$	$C^k_{jk}$		
$R_{ m 33}$	$egin{array}{c} D_{4} \ \dots \end{array}$	$A_{f 3}, \ C^j_{jk}$	$C^j_{kj},\ C^k_{kj},$			$C_l^k$		$C_1$	
	1			$C_2$					
	$D_{10}$	$A_6$ , $C^j_{jk}$	$C^j_{kj},\ C^k_{kj},$			$C_l^k$	$C^k_{jk}$		
$R_{34}$		$\bigcirc jk$		$C_{1}$					
	מ	1		<b>0</b>	Ωi		<i>Ci</i>		
_	$\left  egin{array}{c} D_9 \ D_m \end{array}  ight $	$A_5$ ,	$egin{aligned} C_{k\!j}^j, & C_{k\!j}^k, \ C_{k\!j}^j, & C_{k\!j}^k, \end{aligned}$		$C_r^j \ C_r^j$		$C^j_{jk}$	$C_2$	
$R_{35}$	$\left egin{array}{c} D_{\!\scriptscriptstyle A} \end{array} ight $	$A_2$ ,	$C_{kj}^{j},\;C_{kj}^{j},\;C_{kj}^{k},\;C_{kj}^{k},\;C_{jk}^{k}$		$\circ_r$	$C_l^k$		$C_1$	
		87	$C^k_{jk}$			•		1	
	$\mid D_{9} \mid$	$A_5$ ,	$C^j_{ki}, C^k_{ki}$		$C_r^j$		$C^j_{jk}$		
$R_{ m 36}$	$D_{7}$	$A_8$ ,	$C^k_{jk} \stackrel{C^j_{kj},\ C^k_{kj},}{C^k_{jk}}$		$C^j_{r_{ m I}}$			$C_2$	
		$A_{5}$ , $A_{8}$ , $A_{3}$ , $A_{5}$ , $A_{8}$ ,	$C_{jk}$	$C_1$					
	$D_9$	$A_5$ ,	$C^j_{kj},\; C^k_{kj},$		$C^{j}_{m{r}}$		$C^j_{jk}$		
$R_{ m 37}$	$D_4$	$A_3$ ,	$C^j_{kj},\ C^k_{kj},$			$C_l^k$		$C_1$	
		$A_5$ , $A_3$ , $A_5$ ,	$C^k_{jk}$	$C_2$					
	$D_{9}$	$A_5$ ,	$C^{k}_{jk} $		$C_r^j$		$C^j_{jk}$		
$R_{38}$			$C^k_{jk}$	$C_1$					
				$C_1 \\ C_2$					
$R_{ m so}$	$\mid D_8 \mid$	$A_4$ , $\mathit{C}^{j}_{jk}$ ,	$C^k_{jk}$ , $C^j_{kj},\ C^k_{kj},$		$C_l^j$			$C_2$	
08	$D_4$	$A_3$ ,	$C^j_{kj},\ C^k_{kj},$			$C_l^k$		$C_1$	

TABLE III.—Continued.

(1)	(2)	(8)		Proved by	y (3)	(6)
Nota- tion.	From Table I.	Assumptions.		(4)	(5)	Proved by (3) and (4).
$R_{ m 40}$	$D_3$	$A_4, \ C^{i}_{jk}, \ C^{k}_{jk}, \ A_7, \ C^{i}_{jk}, \ C^{k}_{jk}, \ C^{k}_{jk}, \ C^{k}_{kj}$	$C_l^i \ C_r^k$		$egin{array}{c} C_2 \ C_1 \end{array}$	
$R_{41}$	$egin{array}{c} D_8 \ \dots \ \dots \ \end{array}$	$A_4$ , $C^j_{jk}$ , $C^k_{jk}$ , $C^k_{kj}$ , $C^k_{kj}$ , $C^k_{kj}$ , $C_1$	$C_i^c$		$C_2$	
$R_{42}$	$egin{array}{c} D_7 \ D_4 \ \dots \ \dots \end{array}$	$C_1 \ A_8 , \qquad C^j_{kj}, \ C^k_{kj}, \ A_3 , \qquad C^j_{jk} \ C^k_{kj}, \ C^k_{kj} , \ C^k_{kj} , \ C^k_{jk} $	$C_r^j$ $C_l^k$		$egin{array}{c} C_2 \ C_1 \end{array}$	
$R_{43}$	$egin{array}{c c} D_7 \ D_3 \end{array}$	$egin{aligned} A_8, & C^j_{kj},  C^k_{kj}, \ A_7,  C^j_{jk}, & C^k_{jk}, \end{aligned}$	$C_{m{r}}^{j}$ $C_{m{r}}^{k}$		$egin{array}{c} C_2 \ C_1 \end{array}$	
		$C^{i}_{jk} \ C^{k}_{jk} \ C.$	$C_{r}^{j}$		$C_2$	
- 1	!	$A_3, \qquad \qquad C^j_{jk}, \ C^k_{jk}, \ C^k_{jk} \ \qquad \qquad C^2_2$	$C_l^{k}$		$C_1$	
$R_{46}$	$egin{array}{c} D_3 \ \dots \ \dots \ \end{array}$	$A_{7},\;C_{jk}^{i},\;\;C_{jk}^{k},\;\; \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ $	$C^k_r$		$C_1$	

III.—The following thirty-two different ways of defining reducibility are based on assumptions concerning division in the set E and its subsets,  $E_j$ ,  $E_k$ , simultaneously.

TABLE IV.\*.

(1)	<b>(2</b> )		(3)	Proved by	y (3)	(6)
Nota- tion.	From Table I.		Assumptions.	(4)	(5)	Proved by (8) and (4).
	$D_{12}$	$A_{5},\;C^{j}_{jk},\;C^{jk}_{k},$	$C^{i}_{l}$	$C^{j}_{kj}$		
$R_{\scriptscriptstyle 47}$	$D_{11}$	$A_6$ , $C^j_{jk}$ , $C^k_{jk}$ ,	$C_r^k$	$C_{kj}^k$		
	$D_8$	$A_4$ , $C^j_{jk}$ , $C^k_{jk}$ ,	$C_l^j$		$C_{2}$	
	$D_2$	$A_6, \ C^j_{jk}, \ C^k_{jk}, \ A_4, \ C^j_{jk}, \ C^k_{jk}, \ A_2, \ C^j_{jk}, \ C^k_{jk}, \ C^k_{jk$	$C_l$	" "		$C_1$
	$D_{12}$	$A_5,\ C^j_{jk},\ C^k_{jk},$	$C_t^g$	$C_{kj}^{j}$		
$R_{ m 48}$	$D_{11}$	$egin{aligned} A_6, & C^j_{jk}, & C^k_{jk}, \ A_4, & C^j_{jk}, & C^k_{jk}, \end{aligned}$	$C^k_r$	$C_{kj}^k$		
	$D_8$	$A_4, \ C^j_{jk}, \ C^k_{jk},$	$C_l^i$		$C_{2}$	
	$D_1$	$A_7,\ C^j_{jk},\ C^k_{jk},$	$C_r$	" "	•	$C_1$
	$D_{12}$	$A_5,\ C^j_{jk},\ C^k_{jk},$	$C_i^j$	$C_{kj}^{j}$		
D	$D_{11}$	$A_6,\ C^j_{jk},\ C^k_{jk},$	$C_r^k$	$C_{kj}^k$		
$R_{49}$	$D_7$	$A_8$ ,	$C_r^j$	" "		$C_2$
	$D_2$	$A_{5},\ C^{j}_{jk},\ C^{k}_{jk},\ A_{6},\ C^{ij}_{jk},\ C^{k}_{jk},\ A_{8},\ A_{3},\ C^{ij}_{jk},\ C^{k}_{jk},\ C^{k}_{jk},$	$C_l$	" "		$C_1$
	$D_{10}$	$A_{\epsilon}$ . $C^{j}_{ik}$ . $C^{k}_{ik}$	$C_t^j$	$C^j_{kj}$		
$R_{50}$	$D_{11}$	$A_6$ , $C^j_{jk}$ , $C^k_{jk}$ ,	$C_r^k$	$C_{kj}^k$		
	$D_7$	$A_8$ ,	$*C_r^j$	" "		$C_2$
	1	$A_6, \ C^{i}_{jk}, \ C^{k}_{jk}, \ A_8, \ A_7, \ C^{i}_{jk}, \ C^{k}_{jk}, \ C^{k}_{jk},$	$C_r$	<i>ι: ιι</i>		$C_1$
	$D_{12}$	$egin{aligned} A_5, & C^{i}_{jk}, & C^{k}_{jk}, \ A_6, & C^{i}_{jk}, & C^{k}_{jk}, \ A_4, & C^{i}_{jk}, & C^{k}_{jk}, \end{aligned}$	$C_i^{i}$	$C_{kj}^{j}$		
D	$D_{11}$	$A_6$ , $C^j_{jk}$ , $C^k_{jk}$ ,	$C_r^k$	$C_{\it kj}^{\it k}$		
$K_{51}$	$D_6$	$A_4$ , $C^j_{jk}$ , $C^k_{jk}$ ,	$C_l$	"		$C_2$
	$D_4$	$A_3$ ,	$*C_l^k$	" "		$C_{\mathbf{i}}$

<sup>\*</sup> A condition marked with a star is dependent on the other assumptions in the particular definition in which it appears. Thus \* $C_r$  is dependent upon the other assumptions of  $R_{50}$  (§§4-5).

TABLE IV.—Continued.

(1)	(2)		(3)	Proved by	y (3)	(6)
Nota- tion.	From Table I.		Assumptions.	(4)	(5)	Proved by (3) and (4).
		$A_5$ , $C^j_{jk}$ , $C^k_{jk}$ ,	$C_l^q$	$C^{j}_{kj}$		
$R_{ extsf{52}}$	$D_{11}$	$A_6$ , $C^j_{jk}$ , $C^k_{jk}$ ,	$C^k_{m r}$	$C_{kj}^k$		
$n_{52}$	$\mid D_6 \mid$	$A_4$ , $C^j_{jk}$ , $C^k_{jk}$ ,	$C_{l}$	" "		$C_2$
	1	$A_7$ , $C^j_{jk}$ , $C^k_{jk}$ ,	$C^k_{m r}$		$C_1$	
		$A_5$ , $C^i_{jk}$ , $C^k_{jk}$ ,	$C_l^j$	$C_{k_{\mathcal{I}}}^{i}$		
ъ	$D_{11}$	$A_6$ , $C^j_{jk}$ , $C^k_{jk}$ ,	$C^k_r$	$C_{kj}^{k}$		
11.53	$oxedsymbol{D_6}$	$A_4$ , $C^j_{jk}$ , $C^k_{jk}$ ,	$\mathit{C}_{l}$	""		$C_{\mathfrak{s}}$
	$oxedsymbol{D_2}$	$A_3$ , $C^{ij}_{jk}$ , $C^{k}_{jk}$ ,	$C_l$	66 66		$C_1$
	$D_{12}$	$A_5,\ C^j_{jk},\ C^k_{jk},$	$C_l^q$	$C^{j}_{kj}$		
D	$D_{11}$	$egin{aligned} A_6, \ C^{j}_{jk}, \ C^{k}_{jk}, \ A_4, \ C^{j}_{jk}, \ C^{k}_{jk}, \end{aligned}$	$C_r^k$	$C_{kj}^{k}$		
$R_{ extsf{54}}$	$D_6$	$A_4$ , $C^j_{jk}$ , $C^k_{jk}$ ,	$C_{i}$	66 66		$C_2$
	$D_1$	$A_7,\ C^j_{jk},\ C^k_{jk},$	$C_{r}$	46 44		$C_1$
	$D_{12}$	$A_1, \ C^j_{jk}, \ C^k_{jk}, \ A_7, \ C^j_{jk}, \ C^k_{jk}, \ C^$	$C_i^q$	$C_{kj}^{j}$		
n	$D_{11}$	$A_6$ , $C^j_{jk}$ , $C^k_{jk}$ ,	$C^k_r$	$C_{kj}^k$		
$M_{55}$	$D_5$	$A_8$ , $C^j_{jk}$ , $C^k_{jk}$ ,	$C_r$	""		$C_2$
	$D_4$	$A_3$ ,	$C_l^k$	"		$C_1$
	$D_{12}$	$A_5,\ C^j_{jk},\ C^k_{jk},$	$C_i^q$	$C_{kj}^{i}$		
70	$D_{11}$	$egin{aligned} A_6, & C^j_{jk}, & C^k_{jk}, \ A_8, & C^j_{jk}, & C^k_{jk}, \end{aligned}$	$C_r^k$	$C_{kj}^{k}$		
$R_{ exttt{56}}$			$C_{r}$	" "		$C_2$
		$A_7$ , $C^j_{jk}$ , $C^k_{jk}$ ,	$C^k_r$		$C_{1}$	

## EPSTEEN AND LEONARD: On the Definition of

TABLE IV.—Continued.

(1)	(2)		(3)			Proved by	7 (3)	(	6)
Nota- tion.	From Table I.		Assumption	8.		(4)	(5)	Pro by (3) an	ved y d (4).
$R_{\scriptscriptstyle  extsf{K}^{st}}$		$egin{aligned} A_{5}, \ C^{j}_{jk}, \ C^{k}_{jk}, \ C^{k$			$\mathcal{O}_r^k$	$C^{j}_{kj}$ $C^{k}_{kj}$			
	$D_{\rm o}$	$A_{\alpha}, C_{\alpha}^{i}, C_{\alpha}^{k}$		$C_r$		""		$C_1$	$C_2$
$R_{ t 58}$	$egin{array}{c} D_{12} \ D_{11} \ D_{5} \end{array}$	$A_5, \ C^j_{jk}, \ C^k_{jk}, \ A_6, \ C^j_{jk}, \ C^k_{jk}, \ A_8, \ C^j_{jk}, \ C^k_{jk}, \ C^k_{jk$		$C_{r}^{i}$ $C_{r}$	∩k ′ r	$egin{array}{c} C_{kj}^{i} & & & & & & & & & & & & & & & & & & &$			$C_{2}$
	$D_1$	$A_7$ , $C_{jk}$ , $C_{jk}^a$ ,		$C_{r}$	C(k	ıı ıı		$C_1$	~
$R_{ exttt{59}}$	$egin{array}{c c} D_{10} \ D_{9} \ D_{8} \ \end{array}$	$A_{6}, \ A_{5}, \ A_{4}, \ A_{3},$	$C_{kj}^{i},\;C_{kj}^{i},\ C_{kj}^{i},\;C_{kj}^{k},$	$C_r^j$ $ullet C_l^j$	$C_l^k$	$C^j_{jk}$			$C_{2}$
	$egin{array}{c} D_2 \ D_{10} \ D_9 \ D_8 \ \end{array}$	$A_3$ , $A_6$ ,	$C^j_{kj},\ C^k_{kj},$	$C_{l}$	$C_l^k$	i		$C_1$	
$R_{60}$	$egin{array}{c c} D_9 \ D_8 \ D_1 \ \end{array}$	$A_5$ , $A_4$ , $A_7$ ,	$egin{aligned} C_{kj}^j, & C_{kj}^k, \ & & \ C_{kj}^j, & C_{kj}^k, \end{aligned}$	$C_{r}^{j} \ C_{l}^{i} \ C_{r}$		$C^{j}_{jk}$		$C_{1}$	$C_2$
n	$egin{array}{c} D_{10} \ D_{9} \end{array}$	$A_{6}, \ A_{5},$	$egin{aligned} C_{kj}^j, & C_{kj}^k, \ C_{kj}^j, & C_{kj}^k, \end{aligned}$	$C_{m{r}}^{j}$	$C_l^k$	$egin{array}{c} C_{jk}^k \ C_{jk}^j \end{array}$	·		
$\kappa_{61}$	$egin{array}{c c} D_9 \ D_7 \ D_2 \ \end{array}$	$A_8, \ A_3,$	$egin{aligned} C_{kj}^j, & C_{kj}^k, \ C_{kj}^j, & C_{kj}^k, \end{aligned}$	$C_r^j \ C_l$		"	$C_{2}$	$C_1$	

TABLE IV.—Continued.

(1)	(2)		(3)			Proved	l by (8)	(6)
Nota- tion.	From Table I.		Assumption	ıs.		(4)	(5)	Proved by (3) and (4).
	$D_{10}$	$A_6$ ,	$C^j_{kj},\ C^k_{kj},$		$C_l^k$	$C^k_{jk}$		
$R_{62}$	i i	$A_5$ ,	$C^j_{kj},\ C^k_{kj},$	$C^j_{\ m{r}}$		$C^{j}_{jk}$		
<b></b>	$D_7$	$A_8$ ,	$C^j_{kj},\ C^k_{kj}$ ,	$C^j_{m{r}}$			$C_2$	
	$D_1$	$A_7$ ,	$C^j_{kj},\ C^k_{kj},$	$C_{r}$		66 66		$C_1$
	1 1	$A_6$ ,	$C^j_{kj},\ C^k_{kj},$		$C_{l}^{k}$	$C^k_{jk}$		
$R_{\rm 63}$	1 1	$A_5$ ,	$C_{kj}^j,\ C_{kj}^k,$	$C^j_{m{r}}$		$C^{j}_{jk}$		
•	$\mid D_6 \mid$	$A_4$ ,	$C^{j}_{kj},\ C^{k}_{kj},$	$C_l$		" "		$C_2$
	$D_4$	$A_3$ ,	$C^j_{kj},\;C^k_{kj},$		$C_l^k$		$C_1$	
	$D_{10}$	$A_6$ ,	$C^j_{kj},\ C^k_{kj},$		$C_l^k$	$C^k_{jk}$		
$R_{64}$	$D_{9}$	$A_5$ ,	$C^j_{kj}$ , $C^k_{kj}$ ,	$egin{array}{c} C_{r}^{j} \ C_{l} \end{array}$		$C^{j}_{jk}$		
012	$D_6$	$A_4$ ,	$C_{kj}^i,\ C_{kj}^k,$	$C_l$				$C_{2}$
	$D_3$	$A_7$ ,			$C^k_r$	""		$C_1$
	$D_{10}$	$A_6$ ,	$C^j_{kj},\ C^k_{kj},$		$C_l^k$	$C^k_{jk}$		
n	$D_9$	$A_5$ ,	$C^j_{kj},C^k_{kj},$	$C^j_{m{r}}$		$C^{j}_{jk}$		
$R_{65}$	$D_6$	$A_4$ ,	$C^j_{kj},\ C^k_{kj},$	$C_l$		""		$C_2$
	$D_2$	$A_3$ ,	$C^j_{kj},\ C^k_{kj},$	$C_l$				$C_1$
	$D_{10}$	$A_6$ ,	$C^j_{kj},\ C^k_{kj},$		$C_l^k$	$C^k_{jk}$		
T	1	$A_5$ ,	$C^j_{kj},\;C^k_{kj},$	$C_r^j$		$igg  C^j_{jk}$		
$R_{66}$	$D_6$	$A_4$ ,	$C^j_{kj},\ C^k_{kj},$	$C_{l}$		""		$C_{2}$
	$D_1$	$A_7$ ,	$C^j_{kj},\ C^k_{kj},$	$C_{r}$				$C_1$

TABLE IV.—Continued.

(1)	(2)	(3)			Proved	(6)		
Nota- tion.	From Table I.					(4)	(5)	Proved by (3) and (4).
-	$D_{10}$	$A_6$ ,	$C^j_{kj},\ C^k_{kj},$		$C_l^k$	$C^k_{jk}$		
$R_{ m 67}$	$D_9$	$A_5$ ,	$C^j_{kj},\ C^k_{kj},$	$C^j_{m{r}}$		$C^{j}_{jk}$		
0,	$D_{5}$	$A_8$ ,	$C^j_{kj},\ C^k_{kj},$	$C_{r}$		"		$C_2$
	$D_4$	$A_3$ ,	$C^j_{kj},\ C^k_{kj},$		$C_l^{m{k}}$		$C_1$	
	1	-	$C^j_{kj},\ C^k_{kj},$		$C_l^k$	i		
$R_{68}$	$D_{9}$	$A_5$ ,	$C^j_{kj},\ C^k_{kj},$	$C^{j}_{m{r}}$		$C^{j}_{jk}$		
2068	$D_{5}$	$A_8$ ,	$C^j_{kj},\ C^k_{kj},$	$C_r$				$C_2$
	$D_3$	$A_7$ ,			$^*C^k_{r}$			$C_1$
	1	$A_6$ ,	$C^j_{kj},\ C^k_{kj},$		$C_l^k$	$C^k_{jk}$		
$R_{ m 69}$	$D_{9}$	$A_5$ ,	$C^j_{kj},\ C^k_{kj},$	$C_{m{r}}^{j}$		$C^{j}_{jk}$		
	$D_{b}$		$C^j_{kj},\ C^k_{kj},$	$C_{r}$		· · · · ·		C <sub>2</sub>
	$D_2$		$C^j_{kj},\ C^k_{kj},$	$\mathit{C}_{l}$		« 11		$C_1$
	$D_{10}$	$A_6$ ,	$C^j_{kj},\ C^k_{kj},$		$C_l^k$	$C^k_{jk}$		
R	$D_{9}$	$A_6$ , $A_5$ ,	$C_{kj}^j,\ C_{kj}^k,$	$C^{j}_{m{r}}$		$C^{j}_{jk}$ ,		
1070	$D_5$	A18,	$C^j_{kj},\ C^k_{kj},$	$C_{r}$		" "		C <sub>2</sub>
	$D_1$	$A_7$ ,	$C^j_{kj},\ C^k_{kj},$	$C_{r}$		"		$C_1$
D	$D_8$	$A_4$ , $C^j_{jk}$ , $C^k_{jk}$	,	*0	vi l		$C_2$	
2071	$D_2$	$A_3$ , $C^j_{jk}$ , $C^k_{jk}$	, $C^j_{kj}$ , $C^k_{kj}$ ,	$C_{l}$			$C_1$	
$R_{ra}$	$D_8$	$A_4$ , $C_{ik}^j$ , $C_{ik}^k$	,	C	vi T		$C_2$	
12	$D_1$	$A_7$ , $C^j_{jk}$ , $C^k_{jk}$	, $C^j_{kj},\ C^k_{kj},$	$C_r$			$C_1$	

TABLE IV.—Continued.

(1)	(2)	(3)				Proved	(6)	
Nota- tion.	From Table I.		(4)	(5)	Proved by (3) and (4).			
_	$D_7$	$egin{aligned} A_8, & & & & & & & & & & & & & & & & & & &$	$C^j_{kj},\ C^k_{kj},$	$C^{j}_{m{r}}$			$C_2$	
$R_{73}$	$D_2$	$A_3,\ C^j_{jk},\ C^k_{jk},$	$C^j_{kj},\ C^k_{kj},$	$C_l$			$C_{1}$	
R.,	$D_7$	$A_8$ ,	$C^j_{kj},\ C^k_{kj},$	$*C_{m{r}}^{j}$			$C_2$	
2074	$D_1$	$A_7,\ C^j_{jk},\ C^k_{jk},$	$C^j_{kj},\ C^k_{kj},$	$C_r$			$C_{\mathbf{i}}$	
$R_{\pi}$	$D_6$	$A_4,\ C^j_{jk},\ C^k_{jk},$	$C^{j}_{kj},\ C^{k}_{kj},$	$\mathit{C}_{i}$			$C_2$	
75	$D_4$	$A_3$ ,	$C^j_{kj},\ C^k_{kj},$		$*C_l^k$		$C_{1}$	
$R_{rs}$	$D_6$	$A_4,\ C^j_{jk},\ C^k_{jk},$	$C^j_{kj},\ C^k_{kj},$	$C_l$			$C_2$	
10	$D_3$	$A_7$ , $C_{jk}$ , $C_{jk}^k$ ,			$C_{r}^{k}$		$C_1$	
$R_{"}$	$D_5$	$A_8$ , $C^j_{jk}$ , $C^k_{jk}$ ,	$C^j_{kj},\ C^k_{kj},$	$C_r$			$C_2$	
11	$D_4$	$A_3$ ,	$C^j_{kj},\ C^k_{kj},$		$C_l^{k}$		$C_{\mathbf{i}}$	
$R_{\pi o}$	$D_{5}$	$m{A}_8$ , $C^j_{jk}$ , $C^k_{jk}$	$C^{j}_{kj},C^{k}_{kj},$	$C_r$			$C_2$	
-318	$D_3$	$A_7$ , $C^j_{jk}$ , $C^k_{jk}$ ,			$*C_r^k$		$C_1$	

#### §4.—Independence Proofs.

The sets of conditions  $R_1, R_2, \ldots, R_{78}$  yield reducibility. It can be seen, however, that there are not enough conditions in any one of these sets to prove the associativity A of the system E. In each case it is necessary to add the conditions  $A_1, A_2$ . The set  $R_1$  with the conditions  $A_1, A_2$ , adjoined, we designate by  $R_1^A$  to indicate that the latter conditions yield both reducibility and associativity. Similarly  $R_2^A \equiv [R_2, A_1, A_2]$ ; ...;  $R_{78}^A \equiv [R_{78}, A_1, A_2]$ .

Table V. contains proofs of the independence of the conditions in each of the sets  $R_1, \ldots, R_{78}$ , with the exception of eight— $R_{50}$ ,  $R_{51}$ ,  $R_{59}$ ,  $R_{68}$ ,  $R_{71}$ ,  $R_{74}$ ,  $R_{75}$ ,  $R_{78}$ —in which the subset division assumptions will be shown later to be redundant.

To prove these independencies we employ the following multiplication tables, for which the borders have been omitted:

 $A_1 \! \left| A_2 \! \left| A_3 \! \left| A_4 \! \left| A_5 \! \left| A_6 \! \left| A_7 \! \left| A_8 \! \left| C_1 \! \left| C_2 \! \left| C_{jk}^j \right| C_{jk}^k 
ight| \right. 
ight.$  $|C_r|C_l^j|C_l^k|C_r^j|C_r^k|$  $C_l$ Proof. I. i $\star$  $\star$ ⋆  $\star$  $\star$  $\star$ \*  $\star$ \*  $\star$ 1  $\star$  $\star$  $\star$  $\star$  $\star$  $\star$  $\star$  $\star$  $\star$ ⋆ \*  $\star$  $\star$  $\star$  $\star$  $\star$  $\star$ \*  $\star$  $\star$  $\star$  $\star$  $\star$ `★ Interchange j, k in (1).  $\star$ \*  $\star |\star| \star$ 3  $i_2$  $\star$ \*  $\star$  $\star$  $\star$  $\star$  $\star$  $\star$  $\star$  $\star$ II.  $\star$ \*  $\star$  $\star$  $\star$  $\star$  $\star$ \* Interchange j, k in (8). 4  $\star$ \*  $\star$  $\star$  $\star$  $\star$  $i_2$  $\star$ \*  $\star$  $i_1$  $\star$  $\star$  $\star$ \*  $\star$ 5  $\star$  $\star$  $\star$  $\star$  $\star$  $\star$  $\star$ i★ III. 6  $\star$  $\star$  $\star$  $\star$  $\star$  $\star$ \*  $\star$ \*  $\star$  $\star$ \*  $\star$ IV.  $\star$ Interchange j, k in (6).  $\star$  $\star$  $\star$  $\star$  $\star$  $\star$ \* ⋆  $i_1$  $\star$ \*  $\star$  $\star$  $\star$  $\star$  $i_1$ Interchange j, k in (5).  $\star$  $\star$ ×  $\star$ \*  $\star$  $\star$ \* \*  $\star$  $\star$  $\star$  $\star$  $\star$  $\star$  $\star$ \* \*  $i_2$  $i_1$ 9 \* \*  $\star$  $\star$  $\star$  $\star$  $\star$  $\star$ ×  $\star$  $\star$  $\star$  $\star$  $\star$ v.  $\star$  $\star$  $\star$  $\star$  $\star$  $\star$ \*  $i_2$ VI.  $\star$  $\star$ \*  $\star$  $\star$  $\star$  $\star$ 10  $\star$ \*  $\star$ ★  $\star$  $\star$  $\star$ Interchange j, k in (9). \*  $\star$  $\star$  $\star$  $\star$  $i_1$  $\star$  $i_2$  $\star$  $\star$  $\star$ \*  $\star$ \* 11 ★  $\star$  $\star$  $\star$  $\star$  $\bigstar$  $|i_2|$  $\star$ **12** ★  $\star$  $\star$  $\star$  $\star$  $\star$  $\star$  $\star$  $\star$ \* \*  $i_1$  $\star$  $\star$  $\star$  $\star$  $\star$ VII.  $i_2$  $i_1$  $\star$  $\star$  $\star$ \* 13  $\star$  $\star$  $\star$  $\star$  $\star$  $\star$  $\star$ Interchange j, k in (12).  $\star$  $\star$  $\star$  $\star$  $i_1$ Interchange j, k in (10).  $\star$  $\star$  $\star$  $\star$  $\star$  $\star$ \*  $\star$  $\star$  $\star$  $\star$ \* 14  $\star$  $\star$  $\star$  $\star$  $\star$  $i_2$ \*  $\star$  $i_1$  $\star$  $\star$  $\star$ **15** ★  $\star$  $\star$ VIII.  $\star$  $\star$  $\star$  $\star$  $\star$  $\star$  $\star$  $\star$  $\star$ 

TABLE V.- (Independence Proofs).

According to  $R_1^A$  the system E is reducible and associative under the assumptions  $A_1$ ,  $A_2$ ,  $A_3$ ,  $A_4$ ,  $C_{jk}^i$ ,  $C_{jk}^k$ ,  $C_{kj}^i$ ,  $C_{kj}^k$ ,  $C_1$ .

 $\star \mid \star \mid i_1$ 

 $\star$ 

 $\star \mid \star$ 

\* \*

 $\star |i_2| \star |\star$ 

 $\star$ 

 $i_1$ 

 $i_2$ 

\*

 $\star |\star| \star$ 

Interchange j, k in (15).

IX.

Interchange j, k in (17).

That  $A_1$  is independent of the other conditions in  $R_1^A$  is shown in row 1 of Table V.

The independence of  $A_2$  is shown in row 2.

 $\star |\star| \star |\star| \star$ 

 $\star |\star| \star |\star| \star |\star| \star$ 

16

**18** ★

17

 $\star |\star| \star$ 

 $\star |\star| \star$ 

\*

 $\star$ 

 $\star$ 

Similarly the independence of any one of the conditions of the others can be seen by looking down its column in Table V. for the square marked i. In some cases, as in column  $A_5$ , the letter i occurs in both row 5 and row 6. In order to prove  $A_5$  independent of the other conditions in  $R_9^A$  we use row 6; but

to prove  $A_5$  independent of the other conditions in  $R_{24}^A$ , row 5 must be used. Likewise there can be no confusion in similar cases—such for example as in row 7 and row 8.

§5.—Dependence proofs 
$$(R_{50}, R_{51}, R_{59}, R_{68}, R_{71}, R_{74}, R_{75}, R_{78})$$
.

1. In  $R_{50}$ \* the assumptions are not independent. We proceed to prove that  $C_r^{ij}$  is a consequence of the others, which are mutually independent.

We know by Table I.,  $D_{12}$ ,  $D_{11}$ , that  $C_{kj}^i$  and  $C_{kj}^k$  are consequences of the assumptions of  $R_{50}$  ( $C_r^j$  being omitted). Table I.  $D_1$ ,  $D_5$  show that  $C_1$  and  $C_2$  are consequences of  $A_7$ ,  $A_8$ ,  $C_{jk}^i$ ,  $C_{jk}^k$ ,  $C_{kj}^i$ ,  $C_{kj}^k$ ,  $C_r$ , and, therefore, can be derived from the above-mentioned independent conditions of  $R_{50}$ . According to the condition  $C_r$ , there exist a J and a K such that

$$(J_1 + K_1)(J + K) = 0$$
, only if  $J_1 = 0 = K_1$ .

Multiplying out, we have in view of  $C_{jk}^i$ ,  $C_{jk}^k$ ,  $C_{kj}^i$ ,  $C_{kj}^k$ , that there exist a J and a K such that

$$J_1J + K_1K = 0$$
, only if  $J_1 = 0 = K_1$ .

By  $C_1$ ,  $C_2$  this may be written  $J_3 + K_3 = 0$ . Multiplying the last equation on the left by J' it appears that there exist a J and a K such that

$$J'J_3 + J'K_3 = 0$$
, only if  $J_1 = 0 = K_1$ .

Since  $J'K_3 = 0$  (by  $C_{jk}^j$ ,  $C_{jk}^k$ ) we conclude there exists a J such that

$$J'J_3 = J'(J_1J) = 0$$
, only if  $J_1 = 0$ .

From  $C_l^i$  it follows that there exists a J such that

$$J_1J = 0$$
, only if  $J_1 = 0$ .

- 2. That  $C_l^k$  is dependent upon the remaining assumptions of  $R_{51}$  can be shown in a similar manner.
- 3. That  $C_l^j$  is dependent upon the remaining assumptions of  $R_{59}$  is seen by interchanging k and j in 2 (i. e. in  $R_{51}$ ).
- 4. That  $C_r^k$  is dependent upon the remaining assumptions of  $R_{68}$  is seen by interchanging j and k in 1 (i. e. in  $R_{50}$ ).

<sup>\*</sup> Also in  $R_{\bullet \bullet}^{A}$ .

<sup>†</sup> The following proofs for 2, 3, 4 may also be employed. In  $R_{so}$  change left to right, obtaining thus  $R_{so}$ . In  $R_{so}$  interchange j and k, obtaining thus  $R_{so}$ .

5. In  $R_{71}$ ,  $C_i^i$  is dependent upon the remaining assumptions. For, (Table I.)  $D_2$ ,  $D_6$  show that  $C_1$  and  $C_2$  are consequences of  $A_3$ ,  $A_4$ ,  $C_{jk}^i$ ,  $C_{jk}^k$ ,  $C_{kj}^i$ ,  $C_{kj}^k$ ,  $C_{kj}^i$ ,  $C_{kj}^i$ ,  $C_{kj}^i$ (cf.  $R_1$ ).  $C_i$  means that there exist a J and a K such that

$$(J+K)(J_1+K_1)=0$$
, only if  $J_1=0=K_1$ .

Multiplying out, we have in view of  $C_{ik}^j$ ,  $C_{ik}^k$ ,  $C_{ki}^j$ ,  $C_{ki}^k$ , that

$$JJ_1 + KK_1 = 0$$
, only if  $J_1 = 0 = K_1$ .

Multiplying on the left by J', in view of  $A_4$ ,  $C_{jk}^{i}$ ,  $C_{jk}^{k}$ , it appears that there exists a J such that

$$J'(JJ_1) = 0$$
, only if  $J_1 = 0$ ,

(or, only if  $JJ_1=0$ ). This is  $C_1^j$ .

- That  $C_r^i$  is dependent upon the remaining assumptions of  $R_{74}$  (cf.  $R_5$ ) is seen by changing left to right in  $R_{71}$ .
- 7. That  $C_i^k$  is dependent upon the remaining assumptions of  $R_{75}$  is seen by interchanging j and k in  $R_{71}$  (cf.  $R_1$ ).
- 8. That  $C_r^k$  is dependent upon the remaining assumptions of  $R_{78}$  is seen by interchanging j and k in  $R_{74}$  (cf.  $R_5$ ).

#### §6.—Semireducibility.

An associative hypercomplex number system E, containing a modulus, is said to be semireducible of the first kind (Transactions, vol. 4 (October, 1903), pp. 437-444) when the following conditions are fulfilled:

$$A, C_{jk}^{ij}, C_{kj}^{ij}, C_{1}, C_{2}, C_{r}, C_{l}.$$
 (7)

In this body of conditions, each one is independent of all the others.\* In

$$A, C_{ki}^i, C_2, C_r, C_l \tag{7a}$$

 $A,\ C_{kl}^i,\ C_2,\ C_r,\ C_l$  are satisfied, the group G of the system E will be reducible and will take the form

$$\begin{aligned} x'_{j_3} &= \sum_{j_1} \{ \sum_{j_2} \gamma_{j_1 j_2 j_3} y_{j_2} + \sum_{k_2} \gamma_{j_1 k_2 j_3} y_{k_2} \} x_{j_1}, \\ x'_{k_3} &= \sum_{j_1} \{ \sum_{j_2} \gamma_{j_1 j_2 k_3} y_{j_2} + \sum_{k_2} \gamma_{j_1 k_2 k_3} y_{k_2} \} x_{j_1} + \sum_{k_1} \{ \sum_{j_2} \gamma_{k_1 j_2 k_3} y_{j_2} + \sum_{k_2} \gamma_{k_1 k_2 k_3} y_{k_2} \} x_{k_3}, \\ (j &= 1, \ldots, m; \ k = m + 1, \ldots, n). \end{aligned}$$

The above-mentioned definition of Molien is the special case obtained when  $\gamma_{iki} = 0$  ( $C_{ik}^{i}$ ). My definition of semireducibility of the first kind is the special case obtained from Molien's "nicht ursprüngliche" Systems when  $\gamma_{jjk} = 0$  ( $C_1$ ). Since the conditions (7) are mutually independent, the Molien conditions, which are obtained by omitting  $C_1$  from (7) must be independent and the conditions (7a) which are obtained from Molien's by omitting  $C_{in}^{i}$  must evidently be independent also.

SAUL EPSTEEN

<sup>\*</sup>At the time this definition was framed I was not aware that a more general case had been studied by Molien in the Mathematische Annalen, vol. 41 (1893), pp. 92-93. When E satisfies the above conditions with C, omitted, it has according to Molien an accompanying system. Professor E. H. Moore has recently pointed out to me that when the conditions

order to make the independence proofs we add the following systems to those of §4.

	X.			X	I.
$e_1$	$e_2$	$e_3$	$e_1$	$e_2$	$e_3$
$e_2$	$e_3$	0			$e_3$
$e_3$	0	0	$e_3$	$e_3$	0
$E_j = e$	$_{1},e_{2};E$	$Z_k = e_3$ .	$E_j =$	$e_1; E_k$	$=e_2, e_3.$

A	$oxed{C^{j}_{jk}}$	$C_{kj}^{j}$			$C_r$	$C_l$	Proof.	
$oldsymbol{i}$	*	*	*	*	*	*	I.	
*	i	*	*	*	*	*	VI.	
*	*	$m{i}$	*	*	*	*	VII. interchanging $j$ and $k$ .	
*	*	*	i	*	*	*	X.	
*	*	*	*	i	*	*	XI.	
*	*	*	*	*	i	*	IX.	
*	*	*	*	*	*	i	VIII.	

§ 7.

It was shown in §1 that a hypercomplex number system is reducible when by a proper choice of the units it can be brought to the form

$$E \equiv E_j E_k = e_1 \cdot \ldots \cdot e_m e_{m+1} \cdot \ldots \cdot e_n,$$

where the following conditions are fulfilled:

- A), associativity;
- $(C_{jk})$ ,

 $e_j e_k = 0,$  i. e. every  $\gamma_{jki} = 0;$ 

 $e_k e_j = 0$ ,  $C_{kj}$ ),

i. e. every  $\gamma_{kji} = 0$ ;

- $C_r$ ), right-hand division possible; or
- $C_l$ ), left-hand division possible.

In other words, the conditions

$$(C_1), \qquad e_{j_1}e_{j_2} = \sum_{j_3} \gamma_{j_1j_2j_3}e_{j_3} \qquad (\gamma_{jjk} = 0),$$

$$C_2), \qquad e_{k_1} e_{k_2} = \sum_{k_3} \gamma_{k_1 k_2 k_3} e_{k_3} \quad \left( \gamma_{kkj} = 0 
ight),$$

are both consequences of A,  $C_{ik}$ ,  $C_{ki}$ ,  $C_r$  (or  $C_l$ ).

Suppose now that  $C_{jk}$ ,  $C_{kj}$  are fulfilled, but  $C_l$  and  $C_r$  are not; in this case we have no reason to conclude that every  $\gamma_{jjk}$  and every  $\gamma_{kkj}$  is zero.\* Nevertheless it can be demonstrated that there exist two hypercomplex number systems

$$F_j \equiv f_1 \ldots f_m, F_k = f_{m+1}, \ldots, f_n,$$

such that

$$f_{j_1}f_{j_2} = \sum_{j_3} \gamma_{j_1j_2j_3}f_{j_3}, f_{k_1}f_{k_2} = \sum_{k_3} \gamma_{k_1k_2k_3}f_{k_3}.$$

In particular, when either  $C_l$  on  $C_r$  is fulfilled, we obtain  $f_i = e_i$  (i = 1, ..., n).

In proof we have from the associativity condition

$$(e_{j_1}e_{j_2}) e_{j_4} = e_{j_1}(e_{j_2}e_{j_4}),$$

$$\sum_{i_3=1}^n (\gamma_{j_1 j_2 i_3} \gamma_{i_3 j_4 i_5} - \gamma_{j_2 j_4 i_3} \gamma_{j_1 i_3 i_5}) = 0,$$
 (8)

$$(j_1, j_2, j_4 = 1, \ldots, m; i_5 = 1, \ldots, n).$$

By  $C_{ik}$  and  $C_{ki}$  (7) becomes

$$\sum_{j_{s}=1}^{m} (\gamma_{j_{1}j_{2}j_{s}}\gamma_{j_{2}j_{4}i_{5}} - \gamma_{j_{2}j_{4}i_{5}}\gamma_{j_{1}j_{2}i_{5}}) = 0,$$

$$(j_{s}, j_{s}, j_{s} = 1, \dots, m; i_{s} = 1, \dots, n).$$
(9)

<sup>\*</sup> As an example of this possibility consider the system  $e_1+e_2=0$  0 0. It can easily be verified that the conditions A,  $C_{ik}$ ,  $C_{ki}$  are fulfilled, but clearly  $\gamma_{112}\equiv 1\ (\pm\ 0)$ .

The  $m^3n$  equations (8) include the  $m^4$  equations

$$\sum_{j_3=1}^{m} (\gamma_{j_1 j_2 j_3} \gamma_{j_3 i_4 j_5} - \gamma_{j_2 i_4 j_3} \gamma_{j_1 j_3 i_5}) = 0,$$

$$(j_1, j_2, j_4, j_5 = 1, \dots, m),$$
(10)

and this shows that there exists a system in m units

$$F_j = f_1 \cdot \dots \cdot f_m$$
 such that  $f_{j_1} f_{j_2} = \sum_{j_3} \gamma_{j_1 j_2 j_3} f_{j_3}$ .

By interchanging j and k it follows that there exists a system in m-n units

$$F_k = f_{m+1} \cdot \dots \cdot f_n$$
 such that  $f_{k_1} f_{k_2} = \sum_{k_3} \gamma_{k_1 k_2 k_3} f_{k_3}$ .

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